

CONTACT STRUCTURES ON G_2 -MANIFOLDS AND SPIN 7-MANIFOLDS

M. FIRAT ARIKAN, HYUNJOO CHO, AND SEMA SALUR

ABSTRACT. We show that there exist infinitely many pairwise distinct non-closed G_2 -manifolds (some of which have holonomy full G_2) such that they admit co-oriented contact structures and have co-oriented contact submanifolds which are also associative. Along the way, we prove that there exists a tubular neighborhood N of every orientable three-submanifold Y of an orientable seven-manifold with spin structure such that for every co-oriented contact structure on Y , N admits a co-oriented contact structure such that Y is a contact submanifold of N . Moreover, we construct infinitely many pairwise distinct non-closed seven-manifolds with spin structures which admit co-oriented contact structures and retract onto co-oriented contact submanifolds of co-dimension four.

1. INTRODUCTION

Given a smooth manifold M of dimension seven, one can ask the existence of two different structures, namely, a G_2 structure and a contact structure: We say that M admits a G_2 -structure if the structure group of TM can be reduced to G_2 , which is the exceptional Lie group of all linear automorphisms of the imaginary octonions $im\mathbb{O} \cong \mathbb{R}^7$ preserving a certain cross product. On the other hand, a (co-oriented) contact structure on M is a co-oriented 6-plane distribution which is totally non-integrable. The induced geometries on M have quite different properties. For instance, G_2 -structure determines a unique metric on M , and so there are local invariants in G_2 geometry. Whereas in contact geometry there are no local invariants because any point has a standard neighborhood by Darboux's theorem. It is known that there are 7-manifolds admitting both G_2 and a contact structures, and that these structures are compatible in a certain way [1]. Here we present new examples of such 7-manifolds with the absence of compatibility.

We can describe G_2 geometry more precisely in the following way: Identify the group G_2 as the subgroup of $GL(7, \mathbb{R})$ which preserves the 3-form

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

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where (x_1, \dots, x_7) are the coordinates on \mathbb{R}^7 and $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$. As an equivalent definition, a manifold with a G_2 -structure φ is a pair (M, φ) , where φ is a 3-form on M , such that $(T_p M, \varphi)$ is isomorphic to $(\mathbb{R}^7, \varphi_0)$ at every point p in M . Such a φ defines a Riemannian metric g on M . We say φ is *torsion-free* if $\nabla \varphi = 0$ where ∇ is the Levi-Civita connection of g . The latter holds if and only if $d\varphi = d(*\varphi) = 0$ where “ $*$ ” is the Hodge star operator defined by the metric g . A Riemannian manifold with a torsion free G_2 -structure is called a G_2 -manifold. Equivalently, the pair (M, φ) is called a G_2 -manifold if its holonomy group (with respect to g) is a subgroup of G_2 . Finally, a three dimensional submanifold Y^3 of a manifold M with (torsion-free) G_2 -structure φ is said to be *associative* if φ is a volume form on Y . More details can be found in [3], [4], [7] and [8].

Contact structures are defined in any odd dimension $2n + 1$ for $n \geq 1$. Here we only consider co-oriented contact structures: A *contact form* on a smooth $(2n + 1)$ -dimensional manifold M is a 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$ (i.e., $\alpha \wedge (d\alpha)^n$ is a volume form on M). The *Reeb vector field* of a contact form α is defined to be the unique global nowhere-zero vector field R on M satisfying the equations

$$(1) \quad \iota_R d\alpha = 0, \quad \alpha(R) = 1$$

where “ ι ” denotes the interior product. The hyperplane field (of rank $2n$) $\xi = \text{Ker}(\alpha)$ of a contact form α is called a *(co-oriented) contact structure* on M . The pair (M, ξ) (or sometimes (M, α)) is called a *contact manifold*. A submanifold $Y \subset (M, \xi)$ is said to be a *contact submanifold* if $TY \cap \xi|_Y$ defines a contact structure on Y . We say that two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are *contactomorphic* if there exists a diffeomorphism $f : M_1 \rightarrow M_2$ such that $f_*(\xi_1) = \xi_2$. Also a *strict contactomorphism* between two contact manifolds (M_1, α_1) and (M_2, α_2) is a contactomorphism $f : M_1 \rightarrow M_2$ such that $f^*(\alpha_2) = \alpha_1$.

The equation (1) implies that the Reeb vector field R co-orient ξ and, as a result, the structure group of the tangent frame bundle can be reduced to $U(n) \times 1$. Such a reduction of the structure group is called an *almost contact structure* on M . By a result of [1], every 7-manifold with spin structure (and hence every manifold with G_2 -structure) admits an almost contact structure.

As an alternative but equivalent definition, for R, α and ξ as above, the triple (J, R, α) is called an *associated almost contact structure* for ξ if J is $d\alpha$ -compatible almost complex structure on ξ . Furthermore, if g is a metric on M satisfying

$$g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y) \quad \text{and} \quad d\alpha(X, Y) = g(JX, Y)$$

for all $X, Y \in TM$, then it is called an *associated metric* or sometimes *contact metric*. We refer the reader to [2], [5] for more on contact geometry.

Now suppose that (M, φ) is a manifold with G_2 -structure with an associative submanifold Y . One interesting question is: Does M admit a contact structure η such that Y is a contact submanifold of (M, η) ? Another reasonable one is: Given a contact structure ξ on Y can we extend it to some contact structure η on M so that (Y, ξ) is a contact submanifold of (M, η) ?

In this paper, we will show that both questions above have positive answers by constructing such manifolds. As we will see in Section 3, one can extend any contact structure on Y to some contact structure on its tubular neighborhood in M , and using this fact and a result of [11] we obtain co-oriented contact structures on non-closed G_2 -manifolds having co-oriented contact submanifolds which are also associative. Before studying the G_2 case we will first have similar extension and existence results for spin 7-manifolds and manifolds with G_2 -structures.

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2. PREPARATORY RESULTS

To be able show the existence of contact structures on certain family of spin 7-manifolds, we first need some contact geometry preparation.

Lemma 2.1. *Let Y be any smooth 3-dimensional manifold. Then for any co-oriented contact structure ξ on Y , there exists a co-oriented contact structure η on $Y \times \mathbb{R}^4$ such that $Y \times \{0\}$ is a contact submanifold of $(Y \times \mathbb{R}^4, \eta)$.*

Proof. Since ξ is co-oriented, there is a contact form α on Y such that $\xi = \text{Ker}(\alpha)$. Also consider the 1-form $\lambda = x_1 \wedge dy_1 + x_2 \wedge dy_2$ on \mathbb{R}^4 with the standard coordinates (x_1, y_1, x_2, y_2) . Let $p_1 : Y \times \mathbb{R}^4 \rightarrow Y$ and $p_2 : Y \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$ denote the usual projections. Now consider the pullback forms on $Y \times \mathbb{R}^4$ given by $\tilde{\alpha} := p_1^*(\alpha)$, $\tilde{\lambda} := p_2^*(\lambda)$ and set

$$\beta = \tilde{\alpha} + \tilde{\lambda}.$$

Then $d\beta = d\tilde{\alpha} + d\tilde{\lambda}$. Also using the facts $(d\tilde{\alpha})^i = 0, \forall i \geq 2, (d\tilde{\lambda})^j = 0, \forall j \geq 3$ we compute

$$\beta \wedge (d\beta)^3 = 3 \tilde{\alpha} \wedge d\tilde{\alpha} \wedge (d\tilde{\lambda})^2.$$

Therefore, $\beta \wedge (d\beta)^3$ is a volume form on $Y \times \mathbb{R}^4$ because $\alpha \wedge d\alpha$ and $(d\lambda)^2 = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$ are volume forms on Y and \mathbb{R}^4 , respectively. Hence, β is a contact form which defines a co-oriented contact structure η on $Y \times \mathbb{R}^4$.

For the last statement, consider the embedding $f : Y \rightarrow f(Y) \subset Y \times \mathbb{R}^4$ given by $y \mapsto (y, 0)$ of $Y \times \{0\}$ in $Y \times \mathbb{R}^4$. Then we compute

$$f^*(\beta) = f^*(\tilde{\alpha}) + f^*(\tilde{\lambda}) = f^*(p_1^*(\alpha)) + \underbrace{\tilde{\lambda}|_{Tf(Y)}}_{=0} = \underbrace{(p_1 \circ f)^*}_{id}(\alpha) = \alpha$$

which shows that $\phi : (Y, \alpha) \rightarrow (f(Y), \beta|_{TF(Y)})$ is a strict contactomorphism. In particular, $(Y \times \{0\}, \phi_*(\xi))$ (or equivalently, (Y, ξ)) is a contact submanifold of $(Y \times \mathbb{R}^4, \eta)$. \square

Remark 2.2. The above lemma can be proved in a much more general setting. More precisely, if we take Y to be an arbitrary $(2n+1)$ -dimensional manifold ($n \geq 3$) admitting a co-oriented contact structure and \mathbb{R}^4 to be \mathbb{R}^{2m} ($m \geq 2$), then a similar statement is still true, i.e., there exist a co-oriented contact structure on $Y \times \mathbb{R}^{2m}$ such that $Y \times \{0\}$ is a contact submanifold. Since we are interested in dimension seven, the version stated above will be enough for our purposes.

Next, we present a basic homotopy theoretical fact as a lemma below. Recall that a manifold M admits a spin structure if and only if the second Stiefel-Whitney class $w_2(M)$ of M vanishes.

Lemma 2.3. *Let Y be any orientable smooth 3-manifold embedded in an orientable 7-manifold M with a spin structure. Then there exists a tubular neighborhood of Y in M which is trivial.*

Proof. Since 3-dimensional manifold Y is orientable, it is parallelizable [13, 14], in other words, the tangent bundle TY is trivial. Therefore, $w_i(Y) = 0, \forall i > 0$. Let N be the normal bundle of Y in M . Then by the tubular neighborhood theorem, it suffices to show that N is trivial. Note that $TM|_Y = TY \oplus N$ and so the total Stiefel-Whitney classes satisfies the equation

$$w(TM|_Y) = w(TY)w(N).$$

which gives $w(N) = w(TM|_Y)$ as $w(TY) = 1$. Thus, $w_1(N) = w_2(N) = 0$ because M is orientable and spin.

The vector bundle N (of rank 4) over Y is classified by a Gauss map

$$g : Y \rightarrow BO(4)$$

which lifts to a map $f : Y \rightarrow BSpin(4)$ as $w_1(N) = w_2(N) = 0$. Note that the domain of f is the 3-dimensional manifold Y , and so we are allowed to change the codomain $BSpin(4)$ of f by adding k -cells for $k \geq 5$, and hence we can assume that the homotopy groups π_k of the codomain vanish for $k \geq 4$. Call this new codomain X , and so we can rewrite f as

$$f : Y \rightarrow X \quad \text{with} \quad \pi_k(X) = 0, \quad \forall k \geq 4.$$

On the other hand, since $Spin(4) = SU(2) \times SU(2) = S^3 \times S^3$, its lowest nontrivial homotopy group is $\pi_3 = \mathbb{Z} \times \mathbb{Z}$. Therefore, the lowest nontrivial homotopy group of $BSpin(4)$ is $\pi_4 = \mathbb{Z} \times \mathbb{Z}$ which was already killed in the construction of X above. We conclude that the codomain X is contractible. As a result, $f : Y \rightarrow X$ (and so $g : Y \rightarrow BO(4)$) is homotopically trivial. Equivalently, N is a trivial vector bundle. \square

3. MAIN RESULTS

In this section we will first focus on spin 7-manifolds and manifolds with G_2 -structures, and then we will prove our two main results which can be stated as follows:

Theorem 3.1. *There exist infinitely many pairwise distinct non-closed G_2 -manifolds such that they admit co-oriented contact structures and have co-oriented contact submanifolds which are also associative. Furthermore, the restrictions of the G_2 metrics on the submanifolds are associated metrics of the contact structures on the submanifolds.*

Theorem 3.2. *There are non-closed G_2 -manifolds with holonomy exactly G_2 such that they admit co-oriented contact structures and have co-oriented contact submanifolds which are also associative. Furthermore, the restrictions of the G_2 metrics on the submanifolds are associated metrics of the contact structures on the submanifolds.*

Using the results of the previous section we immediately conclude that

Theorem 3.3. *Let Y be any orientable smooth 3-manifold embedded in an orientable 7-manifold M with a spin structure. Then for any co-oriented contact structure ξ on Y , there is a co-oriented contact structure η on some tubular neighborhood N of Y in M such that (Y, ξ) is a contact submanifold of (N, η) .*

Proof. We know, by Lemma 2.3, that a small enough tubular neighborhood, say N , of Y in M is trivial. Therefore, we may write $N \approx Y \times \mathbb{R}^4$. (note that $Y \subset N$ is identified with $Y \times \{0\} \subset Y \times \mathbb{R}^4$). On the other hand, Lemma 2.1 implies that for any co-oriented contact structure ξ on Y , there exists a co-oriented contact structure η on $Y \times \mathbb{R}^4$ (and so on N) such that (Y, ξ) is a contact submanifold of $(N \approx Y \times \mathbb{R}^4, \eta)$. □

Since every manifold with G_2 -structure admits a spin structure, we have the following corollary:

Corollary 3.4. *Let Y be any orientable smooth 3-manifold embedded in a manifold M with G_2 -structure. Then for any co-oriented contact structure ξ on Y , there exists a co-oriented contact structure η on some tubular neighborhood N of Y in M such that (Y, ξ) is a contact submanifold of (N, η) .* □

Now observe that with a little more care in the proof of Lemma 2.3, we can see that a trivial neighborhood of an orientable 3-manifold in an orientable spin 7-manifold is itself a manifold with a spin structure. Indeed, we can also proceed as follows: Let Y be any closed orientable smooth 3-manifold and consider the product $M = Y \times \mathbb{R}^4$. Then since $TM = TY \times T\mathbb{R}^4$ and from the fact that $w_2(Y) = 0$ (Y is parallelizable), we have $w_2(M) = 0$. Therefore, M admits a spin structure. On the other hand, Y admits a co-oriented contact

structure ξ [9], [10], [15]. Also we know by Lemma 2.1 that for any co-oriented contact structure ξ on Y we can construct a co-oriented contact structure on $Y \times \mathbb{R}^4$ such that (Y, ξ) is a contact submanifold. Also note that distinct (non-homeomorphic) Y 's gives distinct products. We can summarize this paragraph as the following theorem.

Theorem 3.5. *There exist infinitely many pairwise distinct non-closed 7-manifolds with spin structure such that they admit co-oriented contact structures and retract onto co-oriented contact submanifolds of co-dimension four.* \square

Proof of Theorem 3.1. We start our proof by recalling a useful theorem proved in [11] which is, in fact, the key result for the proof.

Theorem 3.6 ([11]). *Assume (Y^3, g) is a closed, oriented, real analytic Riemannian 3-manifold. Then there exists a G_2 -manifold (M^7, φ) and an isometric embedding $i : Y \rightarrow M$ such that the image $i(Y)$ is an associative submanifold of M . Moreover, (M, φ) can be chosen so that $i(Y)$ is the fixed point set of a nontrivial G_2 -involution $r : M \rightarrow M$. Moreover, as long as the metric g on Y is not flat, the holonomy of M is exactly G_2 .*

Let Y be any closed, oriented, real analytic 3-manifold. Then as before we know [9], [10], [15] that there exists a co-oriented contact structure ξ on Y . Let g be a metric on Y associated to ξ . Then applying Theorem 3.6 to the Riemannian manifold (Y, g) , we know that there exists a G_2 -manifold (M, φ) such that Y is an associative submanifold and the G_2 -metric g_φ restricts to g on Y . On the other hand, the construction used in the proof of Theorem 3.6 also implies that M is, indeed, topologically a thickening of Y (in other words, topologically $M \approx Y \times \mathbb{R}^4$). Therefore, by Lemma 2.1, there exists a co-oriented contact structure η on $M \approx Y \times \mathbb{R}^4$ such that (Y, ξ) is a contact submanifold of (M, η) .

Moreover, clearly we have infinitely many distinct choices for the 3-manifold Y , and so we can construct infinitely many pairwise distinct G_2 -manifolds with the properties described above. This finishes the proof. \square

Proof of Theorem 3.2. We basically follow the same steps used in the previous proof, with some exceptions. More precisely, since we want to construct G_2 -manifolds with holonomy exactly G_2 , the choice of Y is more restrictive. By Theorem 3.6, if the metric g on Y is not flat, then M has holonomy G_2 . So we need to find a closed, oriented, real analytic 3-manifold admitting a contact structure whose associated metric is not flat. To this end, we recall the fact that there exist closed orientable 3-manifolds which admit no flat contact metrics [12]. Therefore, if Y is chosen to be one of these 3-manifolds, then for any contact structure on Y , its associated metric (i.e., contact metric) is not flat. Hence, the result follows. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER NY, USA
 MAX PLANCK INSTITUTE FOR MATHEMATICS, BONN, GERMANY
E-mail address: arikan@math.rochester.edu, arikan@mpim-bonn.mpg.de

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER NY, USA
E-mail address: cho@math.rochester.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER NY, USA
E-mail address: salur@math.rochester.edu